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Automated Derivation of Geometry Theorems

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Abstract

Derivation of geometry theorems belongs to mighty tools of automated geometry theorem proving. By elimination of suitable variables in the system of algebraic equations describing a geometric situation we get required formulas. The power of derivation is presented on computation of the area of planar polygons given by their lengths of sides and diagonals. This part we conclude with derivation of a formula of Robbins for the area of a cyclic pentagon given by its side lengths.
Searching for loci of points of given properties is a special case of derivation. This topic belongs to the most difficult parts of school mathematics all over the world. New technologies DGS and CAS enable to overcome this problem. We demonstrate it in a few examples from elementary geometry.

1 Introduction

In the paper we will be concerned with derivation of geometric theorems by automated tools.
First we introduce derivation as a part of the theory of automated geometry theorem proving. We continue with deriving formulas for the area of a quadrilateral and a pentagon in the plane given by their lengths of sides and diagonals. Then the formula of Brahmagupta for the area of a cyclic quadrilateral given by its side lengths is investigated. This part is concluded by derivation of the analogous formula for a cyclic pentagon. Whereas Brahmagupta formula comes from 6th century AD, it took almost 1400 years until 1994 American Robbins found it [12]. In this paper author's specific approach of finding this formula is shown.
The second part of the paper is devoted to a special class of derivation — searching for loci of points of given properties. This topic belongs to the most difficult parts of school mathematics. New technologies such as dynamic geometry systems (DGS) and computer algebra systems (CAS) facilitate this problem. We will show how to search for loci using automated tools. First we use DGS to state a conjecture, then we apply CAS to find the searched locus exactly. The method is suitable for all school levels from elementary schools to universities.
During computations we will use dynamic geometry system GeoGebra, and computer algebra systems CoCoA[1] based on Gröbner bases (GB) computation and Epsilon[2] based on Wu-Ritt (WR) approach. Computations were done on PC Intel Core2 Duo 3.16GHz.

[1]Software CoCoA is freely distributed at the address http://cocoa.dima.unige.it
[2]Software Epsilon is freely distributed at http://www-calfor.lip6.fr/~wang/epsilon/

2 Automated derivation

Automatic derivation is a part of automatic discovery of theorems in geometry. Whereas in automatic discovery we search for complementary hypotheses for a geometric statement to become true, by automatic derivation of theorems we mean finding geometric formulas holding among prescribed geometric magnitudes which follow from the given assumptions [10]. Let us say it more precisely.
Denote by $K[x_1,\dots,x_n]$ the ring of polynomials of n indeterminates $x=(x_1,\dots,x_n)$ with coefficients in the field K, where K is a field of characteristic zero, for instance the field of rational numbers. Assume that polynomial equations $h_1(x_1,\dots,x_n)=0,\dots,h_r(x_1,\dots,x_n)=0$ express geometric properties of some objects. Let $x_1,\dots,x_m$ be independent variables (parameters) and $x_{m+1},\dots,x_n$ dependent variables. Eliminating variables (dependent or independent) we get the elimination ideal which contains only polynomials in those variables we did not eliminate. Usually we eliminate independent variables $x_1,\dots,x_m$ or, if needed, some dependent variables $x_{m+1},\dots,x_p$, $m\le p\le n$ to obtain a geometric statement expressed by the equation $c(x_{p+1},\dots,x_n)=0$ which follows from the assumptions $h_1=0,\dots,h_r=0$. The theorem holds [8]:

Theorem: *Let $I=(h_1,\dots,h_r)\subset K[x_1,\dots,x_n]$ and $c\in I\cap K[x_{p+1},\dots,x_n]$, for $p\le n$. Then*

$$h_1(x_1,\dots,x_n)=0,\dots,h_r(x_1,\dots,x_n)=0\Rightarrow c(x_{p+1},\dots,x_n)=0.$$

In the next section we will show several examples on derivation of known or less known formulas from geometry of polygons.

2.1 Area of polygons

We will study the area of a planar polygon $A_1A_2\dots A_n$ which is given by its lengths of sides and diagonals. The (signed) area of a polygon is defined regardless of whether it intersects itself or not. The theorem holds [7]:

Theorem: *Let $d_{ij}=|A_iA_j|^2$ denote a square of the distance of the vertices A_i,A_j. Then the area p of an n-gon $A_1A_2\dots A_n$ is given by*

$$16p^2=\sum_{i,j=1}^{n}\begin{vmatrix} d_{i,j} & d_{i,j+1}\\ d_{i+1,j} & d_{i+1,j+1}\end{vmatrix}. \tag{1}$$

In the following we derive in automated way special cases of the theorem above — formulas for a quadrilateral and a pentagon given by their lengths of sides and diagonals.
For $n=3$ we get the well-known formula of Heron

$$16p^2=-a^4-b^4-c^4+2a^2b^2+2b^2c^2+2c^2a^2. \tag{2}$$

As automated derivation of the Heron's formula is quite frequent in the literature, we omit it.
Let us derive a formula for the area of a quadrilateral:

Consider a planar quadrilateral $ABCD$ with lengths of sides a,b,c,d and diagonals e,f. We are to express the area p of $ABCD$ in terms of a,b,c,d,e,f.

Introduce a rectangular coordinate system such that $A=[0,0]$, $B=[a,0]$, $C=[u,v]$, $D=[w,z]$,

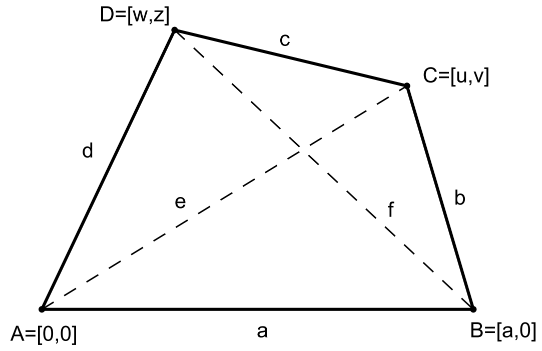

Figure 1: Area of a quadrilateral $ABCD$

Fig. 1. We express relations between b, c, d, e, f, p and coordinates a, u, v, w, z by the following system of algebraic equations:

$b = |BC| \Rightarrow h_1 := (u-a)^2 + v^2 - b^2 = 0,$

$c = |CD| \Rightarrow h_2 := (w-u)^2 + (z-v)^2 - c^2 = 0,$

$d = |DA| \Rightarrow h_3 := w^2 + z^2 - d^2 = 0,$

$e = |EF| \Rightarrow h_4 := u^2 + v^2 - e^2 = 0,$

$f = |EF| \Rightarrow h_5 := (w-a)^2 + z^2 - f^2 = 0,$

$p =$ area of $ABCD \ \Rightarrow h_6 := p - 1/2(av - vw + uz) = 0.$

Elimination of variables u, v, w, z in the system $h_1 = 0$, $h_2 = 0$, $\ldots$, $h_6 = 0$ gives the elimination ideal in variables a, b, c, d, e, f, p. In CoCoA we get

```
Use R::= Q[a,b,c,d,e,f,p,u,v,w,z];
I:=Ideal((u-a)^2+v^2-b^2,(w-u)^2+(z-v)^2-c^2,w^2+z^2-d^2,
u^2+v^2-e^2,(w-a)^2+z^2-f^2,p-1/2(av-vw+uz));
Elim(u..z,I);
```

four polynomials as generators of the corresponding elimination ideal. One of them leads to the equation

$$16p^2 = 4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2 \tag{3}$$

which is the desired result.[3] We can verify that (3) is in accordance with (1) for $n = 4$.

Elimination of u, v, w, z in the system $h_1 = 0$, $h_2 = 0$, $\ldots$, $h_5 = 0$ gives the elimination ideal which is generated by the only polynomial

$M := -2(a^4c^2 - a^2b^2c^2 + a^2c^4 - a^2b^2d^2 + b^4d^2 - a^2c^2d^2 - b^2c^2d^2 + b^2d^4 + a^2b^2e^2 - a^2c^2e^2 - b^2d^2e^2 +$

[3]The remaining three polynomials are also in variables a, b, c, d, e, f, p, and hence express p in terms of a, b, c, d, e, f. They can be derived from (3) and the following relation (4).

$c^2d^2e^2 - a^2c^2f^2 + b^2c^2f^2 + a^2d^2f^2 - b^2d^2f^2 - a^2e^2f^2 - b^2e^2f^2 - c^2e^2f^2 - d^2e^2f^2 + e^4f^2 + e^2f^4)$.

It holds

$$M := \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & e^2 & d^2 \\ 1 & a^2 & 0 & b^2 & f^2 \\ 1 & e^2 & b^2 & 0 & c^2 \\ 1 & d^2 & f^2 & c^2 & 0 \end{vmatrix}.$$

M is the well-known Cayley–Menger determinant [1]. The condition

$$M = 0 \tag{4}$$

expresses a mutual dependence of all six distances between the vertices of a planar quadrilateral $ABCD$.

Similarly we can derive a special case of (1) for a planar pentagon $ABCDE$ with lengths of sides a, b, c, d, e and diagonals i_1, i_2, i_3, i_4, i_5 with the area p. If we denote $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DE|$, $e = |EA|$, and $i_1 = |CE|$, $i_2 = |AD|$, $i_3 = |BE|$, $i_4 = |AC|$, $i_5 = |BD|$, Fig. 2, then

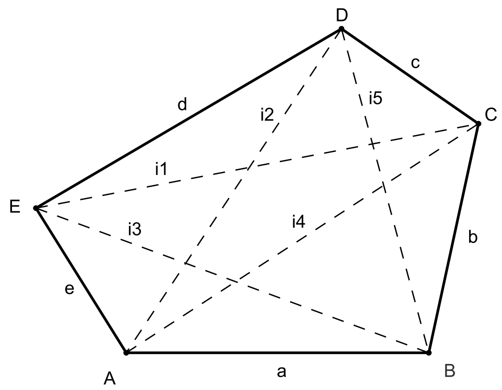

Figure 2: Area of a pentagon

$$\begin{aligned} 16p^2 = {} & -(a^4 + b^4 + c^4 + d^4 + e^4) + 2(a^2b^2 + b^2c^2 + c^2d^2 + d^2e^2 + e^2a^2) \\ & +2(i_1^2i_2^2 + i_2^2i_3^2 + i_3^2i_4^2 + i_4^2i_5^2 + i_5^2i_1^2) - 2(a^2i_1^2 + b^2i_2^2 + c^2i_3^2 + d^2i_4^2 \\ & +e^2i_5^2). \end{aligned} \tag{5}$$

2.2 Area of cyclic polygons

In this part we will study cyclic polygons, i.e., those whose vertices lie on a circle. We will derive area of cyclic polygons in terms of their side lengths. We start from well-known formulas for area of a triangle and a cyclic quadrilateral. This part we conclude with derivation of the formula of Robbins [12] for the area of a cyclic pentagon.

As any triangle is cyclic then the formula for the area of a triangle with side lengths a, b, c is the same as the formula of Heron (2).
The analogy of the formula of Heron for a cyclic convex quadrilateral with side lengths a, b, c, d and the area p is the following formula of Brahmagupta (Brahmagupta — Indian mathematician, 598–c. 665 A.D.)

$$16p^2 = (-a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d). \tag{6}$$

Since that time no formula for a cyclic pentagon with given side lengths a, b, c, d, e and the area p, despite a great effort of mathematicians, appeared until 1994 when American D. P. Robbins [12] discovered it. It took almost 1400 years than the formula for the area of a cyclic pentagon appeared. The reason why it lasted so long is a big complexity of such a formula.
Whereas Robbins combined several methods to discover the formula for a cyclic pentagon, we will demonstrate a method of deriving such a formula based on the theory of automated derivation.

First we will derive formula of Brahmagupta, then we show how to derive the formula of Robbins.

Problem (Brahmagupta): *Given a cyclic quadrilateral with side lengths a, b, c, d and the area p. Find a relation among a, b, c, d, p.*

We will solve the problem using coordinate-free approach. Consider a cyclic quadrilateral $ABCD$ with side lengths a, b, c, d with the area p. Denote by e, f its lengths of diagonals, Fig. 3.

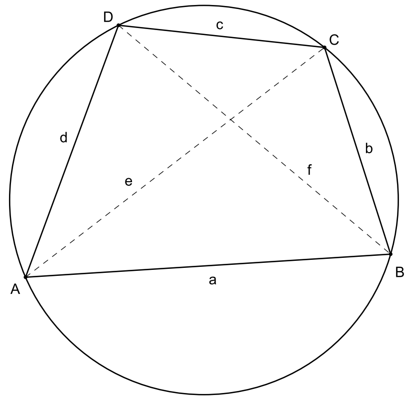

Figure 3: Cyclic quadrilateral $ABCD$

By well-known formulas of Ptolemy [2] we express that $ABCD$ is cyclic. It holds

$$ac + bd - ef = 0 \tag{7}$$

for a cyclic convex quadrilateral and

$$ac - bd + ef = 0 \quad \text{or} \quad -ac + bd + ef = 0$$

for a cyclic non-convex quadrilateral.
First suppose that $ABCD$ is a convex cyclic quadrilateral. Consider the following hypotheses:

p is the area of $ABCD \;\Rightarrow h_1 := 4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2 - 16p^2 = 0$

by the formula (3), and

$ABCD$ is cyclic and convex $\;\Rightarrow h_2 := ac + bd - ef = 0.$

The elimination of variables e, f from the system $h_1 = 0$, $h_2 = 0$ gives

```
Use R ::= Q[a,b,c,d,e,f,p];
I:=Ideal(4e^2f^2-(a^2-b^2+c^2-d^2)^2-16p^2,ac+bd-ef);
Elim(e..f,I);
```

the formula (6).
Now consider that a quadrilateral $ABCD$ is cyclic non-convex. Then

$ABCD$ is cyclic and non-convex $\Rightarrow$

$h_3 := ac - bd + ef = 0$ or $h_4 := -ac + bd + ef = 0.$

The elimination of e, f in the system $h_1 = 0$ and $h_3 = 0$ gives

```
Use R ::= Q[a,b,c,d,e,f,p];
I:=Ideal(4e^2f^2-(a^2-b^2+c^2-d^2)^2-16p^2,ac-bd+ef);
Elim(e..f,I);
```

the formula

$$16p^2 = (-a + b - c + d)(a - b - c + d)(a + b + c + d)(a + b - c - d). \tag{8}$$

The remaining relation $h_4 = 0$ together with $h_1 = 0$ give the same result (8).
Thus for a cyclic quadrilateral with given side lengths a, b, c, d we get *two* formulas (6) and (8) which differ only in one term. Namely if we compute the products in (6) and (8) we get

$$16p^2 = -(a^4 + b^4 + c^4 + d^4) + 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) + 8abcd \tag{9}$$

in a convex case, and

$$16p^2 = -(a^4 + b^4 + c^4 + d^4) + 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) - 8abcd \tag{10}$$

in a non-convex case, Fig. 4.

Note, that both polynomials in (9) and (10) on the right are symmetric polynomials, i.e., by any change of the order of variables a, b, c, d the formulas remain unchanged. Denote by k, l, m, n elementary symmetric functions in variables a^2, b^2, c^2, d^2, i.e.,

$k = a^2 + b^2 + c^2 + d^2,$

$l = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2,$

$m = a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2,$

$n = a^2b^2c^2d^2.$

and let $s = 16p^2$. Then both formulas (9) and (10) can be expressed by one formula

$$(k^2 - 4l + s)^2 - 64n = 0. \tag{11}$$

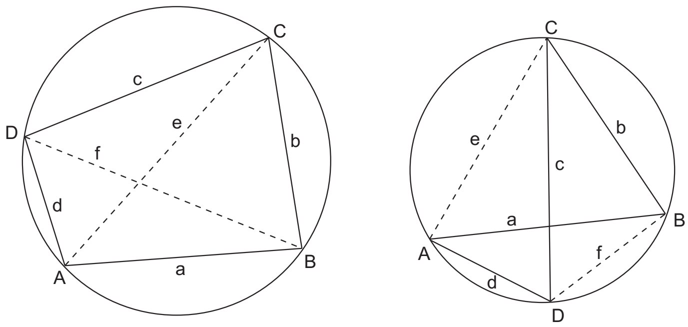

Figure 4: Two cyclic quadrilaterals with the same side lengths a, b, c, d — convex and non-convex cases

Similarly we can express the formula of Heron which reads

$$k^2 - 4l + s = 0, \tag{12}$$

where $k = a^2 + b^2 + c^2$, $l = a^2b^2 + a^2c^2 + b^2c^2$ and $s = 16p^2$.

Remark:

Note the similarity of the formulas (11) and (12). If we put for instance $d = 0$ then a quadrilateral becomes a triangle and the formula (11) becomes (12).

Now we we will derive the formula of Robbins [12], [8].

Problem (Robbins): *Let $ABCDE$ be a cyclic pentagon with side lengths a, b, c, d, e and the area p. Find a relation among a, b, c, d, e, p.*

To solve the problem we will use coordinate-free approach. Consider a cyclic pentagon $ABCDE$ with sides $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DE|$, $e = |EA|$, and diagonals $i_1 = |CE|$, $i_2 = |AD|$, $i_3 = |BE|$, $i_4 = |AC|$, $i_5 = |BD|$, Fig. 5.
First suppose that $ABCDE$ is a convex cyclic pentagon. We will use the formula (5) to express the area p of $ABCDE$ in terms of its lengths of sides a, b, c, d, e and diagonals i_1, i_2, i_3, i_4, i_5. Now we need conditions for a pentagon $ABCDE$ to be cyclic. Using the Ptolemy theorem on cyclic convex quadrilaterals $ABCD$, $BCDE$, $CDEA$, $DEAB$ and $EABC$ we get

$h_1 := ac + bi_2 - i_4i_5 = 0,$

$h_2 := bd + ci_3 - i_5i_1 = 0,$

$h_3 := ce + di_4 - i_1i_2 = 0,$

$h_4 := da + ei_5 - i_2i_3 = 0,$

$h_5 := eb + ai_1 - i_3i_4 = 0.$

Applying the Ptolemy conditions $h_1 = 0$, $h_2 = 0$, $h_3 = 0$, $h_4 = 0$, $h_5 = 0$, to the formula (5) we get the important relation

$$k^2 - 4l + s = 4(eabi_1 + abci_2 + bcdi_3 + cdei_4 + deai_5), \tag{13}$$

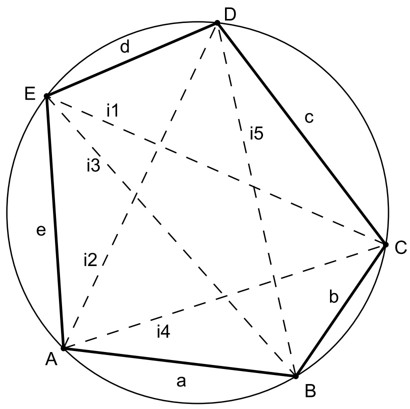

Figure 5: Cyclic pentagon $ABCDE$

where

$k = a^2 + b^2 + c^2 + d^2 + e^2,$

$l = a^2b^2 + a^2c^2 + a^2d^2 + a^2e^2 + b^2c^2 + b^2d^2 + b^2e^2 + c^2d^2 + c^2e^2 + d^2e^2,$

$m = a^2b^2c^2 + a^2b^2d^2 + a^2b^2e^2 + a^2c^2d^2 + a^2c^2e^2 + a^2d^2e^2 + b^2c^2d^2 + b^2c^2e^2 + b^2d^2e^2 + c^2d^2e^2,$

$n = a^2b^2c^2d^2 + a^2b^2c^2e^2 + a^2b^2d^2e^2 + a^2c^2d^2e^2 + b^2c^2d^2e^2,$

$o = a^2b^2c^2d^2e^2$

are elementary symmetric functions and $s = 16p^2$.

Now we need "to get rid" of variables i_1, i_2, i_3, i_4, i_5 in (13) to obtain a formula in a, b, c, d, e and p. To ensure the planarity of $ABCDE$ it suffices to ensure planarity of quadrilaterals $ABCD$, $BCDE$, $CDEA$, $DEAB$ and $EABC$. We remind the relation (4) which is a necessary and sufficient condition for a quadrilateral to be planar. But the use of (4) by elimination of i_1, i_2, i_3, i_4, i_5 is time consuming. Therefore we introduce the following simplification.

If a quadrilateral is cyclic then (4) can be simplified by the following formula [13], [9]:

$$S^2 = PV - 1/2M, \tag{14}$$

where $S = e(ab + cd) - f(bc + ad)$, $P = ac + bd - ef$ and $V = ac(-a^2 - c^2 + b^2 + d^2 + e^2 + f^2) + bd(a^2 + c^2 - b^2 - d^2 + e^2 + f^2) - ef(a^2 + c^2 + b^2 + d^2 - e^2 - f^2)$.

Suppose that $P = 0$. Then (14) implies that instead of the condition $M = 0$ we can take $S = 0$. This gives five conditions for quadrilaterals $ABCD$, $BCDE$, $CDEA$, $DEAB$, $EABC$ to be planar:

$h_6 := i_4(ab + ci_2) - i_5(bc + ai_2) = 0,$

$h_7 := i_5(bc + di_3) - i_1(cd + bi_3) = 0,$

$h_8 := i_1(cd + ei_4) - i_2(de + ci_4) = 0,$

$h_9 := i_2(de + ai_5) - i_3(ea + di_5) = 0,$

$h_{10} := i_3(ea + bi_1) - i_4(ab + ei_1) = 0.$

Now we are ready to derive the formula of Robbins. Let us express the right side of (13) in terms of a, b, c, d, e. Denote

$h_{11} := 4(eabi_1 + abci_2 + bcdi_3 + cdei_4 + deai_5) - t = 0,$

where t is a slack variable.
Now we will eliminate variables i_1, i_2, i_3, i_4, i_5 in the set of polynomials $h_1, h_2, \ldots, h_{11}$. As the Ptolemy polynomials $h_1, h_2, \ldots, h_5$, and similarly the polynomials $h_6, h_7, \ldots, h_{10}$, are dependent, it suffices to consider for instance the ideal I which is generated by six polynomials $h_3, h_4, h_5, h_8, h_9, h_{11}$. CoCoA gives

```
Use R::=Q[a,b,c,d,e,i[1..5],t];
I:=Ideal(ce+di[4]-i[1]i[2],da+ei[5]-i[2]i[3],eb+ai[1]-i[3]i[4],
i[1](cd+ei[4])-i[2](de+ci[4]), i[2](de+ai[5])-i[3](ea+di[5]),
t-4(eabi[1]+abci[2]+bcdi[3]+cdei[4]+deai[5]));
Elim(i[1]..i[5],I);
```

in 1m 4s elimination ideal which is generated by one polynomial in a, b, c, d, e, t with 827 terms. Substitution of elementary symmetric functions k, l, m, n, o and elimination of a, b, c, d, e gives a polynomial equation $Q = 0$ with 37 terms, where

$Q := t^7 + t^6l + t^5km + t^4k^2n + t^3k^3o + t^4m^2 - 12t^5n - 12t^4ln - 8t^3kmn - 8t^2k^2n^2 - 36t^4ko - 36t^3klo - 30t^2k^2mo - 36tk^3no - 27k^4o^2 - 8t^2m^2n + 48t^3n^2 + 48t^2ln^2 + 16tkmn^2 + 16k^2n^3 - 72t^3mo - 72t^2lmo - 96tkm^2o + 144t^2kno + 144tklno - 72k^2mno + 216tk^2o^2 + 216k^2lo^2 + 16m^2n^2 - 64tn^3 - 64ln^3 - 64m^3o + 288tmno + 288lmno - 432t^2o^2 - 864tlo^2 - 432l^2o^2.$

Next substitution $(k^2 - 4l + s) - 4t = 0$, $(k^2 - 4l + s)^2 - 64n - u = 0$, $k(k^2 - 4l + s) + 8m - v = 0$, $128o - w = 0$ together with elimination of k, l, m, n, o, t in the ideal L

```
Use R::=Q[u,v,w,k,l,m,n,o,t,s];
L:=Ideal(Q,k^2-4l+s-4t,(k^2-4l+s)^2-64n-u,k(k^2-4l+s)+8m-v,128o-w);
Elim(k..t,L);
```

gives the final result

$$u^3s + u^2v^2 - 16v^3w - 18uvws - 27w^2s^2 = 0 \tag{15}$$

which is the formula of Robbins.

Similarly we proceed in the case of a non-convex cyclic pentagon. Also in this case we get the same result (15) [12].

Remark:

1. Notice that the formula (15) is of the 7th degree in $s = 16p^2$, where p is the area of a pentagon. This means that there exist at most *seven* cyclic pentagons with given side lengths a, b, c, d, e and different radii.

2. If we put for instance $e = 0$ in the pentagon $ABCDE$ then it becomes a quadrilateral and the formula (15) transforms into the formula (11) for a quadrilateral.

3. As far as I know, the explicit formula for the area of a cyclic n-gon exists for $n = 3, 4, 5, 6, 7, 8$. See [6] for details.

3 Derivation of locus equations

The method of derivation can be also used to determine the locus equations of a motion whose geometric description is given, see [15], [16].
Searching for loci of points forms one part of the geometry seminar which I lead for several years at the University of South Bohemia. The reason why this topic is included into the seminar is natural. In practice we often meet situations in which we are to determine a trajectory of a point by a given motion. Another reason is that searching for loci belongs to the most difficult parts of a school curricula. By searching for loci we keep the following rules:

- First *demonstrate* the problem and construct some points of the searched locus.
- On the base of the previous step try to *guess* the locus.
- Then use the icon Locus in DGS (GeoGebra, Cabri, ...) to *verify* the locus. Remember that this is an exact mathematical proof!
- Using CAS (Derive, Maple Mathematica,...) *derive* the locus equation exactly.

3.1 Loci in plane

To describe derivation of locus equations orderly, we start with the following problem:

Let ABC be a triangle with the given base AB and the vertex C on a line k. Find the locus of the orthocenter G of ABC if C moves on the line k.

First we demonstrate the problem in GeoGebra. When we move the vertex C along the line k we see

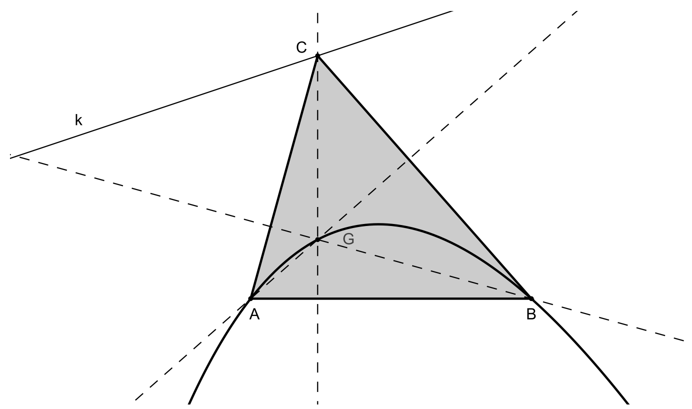

Figure 6: If C moves on k then G moves on a curve similar to parabola

that the orthocenter G moves along the curve which is similar to parabola, Fig. 6. Another position

of the line k gives a curve which is similar to hyperbola, Fig. 7.
We can conclude that the locus is probably hyperbola or parabola.

The question arises:

What is the solution?

To decide this we will derive the locus equation using CAS.

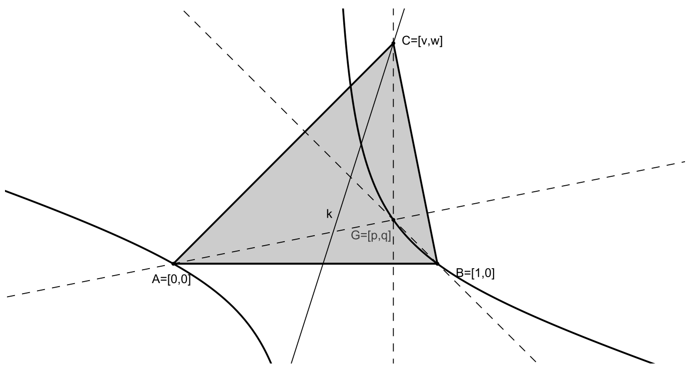

Figure 7: If C moves on k then G moves on a curve similar to hyperbola

Let us place a rectangular coordinate system so that $A = [0, 0]$, $B = [1, 0]$, $C = [v, w]$, $G = [p, q]$ and let k be an arbitrary line with the equation $k : ax + by + c = 0$, Fig. 7. We translate the geometry situation into the following set of polynomial equations:

For the intersection $G = [p, q]$ of heights h_{AB} and h_{BC} it holds:

$G \in h_{AB} \Rightarrow h_1 : p - v = 0$,

$G \in h_{BC} \Rightarrow h_2 : (v - 1)p + wq = 0$.

Further

$C \in k \Rightarrow h_3 : av + bw + c = 0$.

We get the system of three equations $h_1 = 0$, $h_2 = 0$, $h_3 = 0$ in variables a, b, c, v, w, p, q where a, b, c are independent variables, whereas v, w, p, q are dependent variables. To find the locus of $G = [p, q]$ we eliminate variables v, w in the ideal $I = (h_1, h_2, h_3)$ to get a relation in p, q which depends on a, b, c. In CoCoA we enter

```
Use R::=Q[a,b,c,v,w,p,q];
I:=Ideal(p-v,(v-1)p+wq,av+bw+c);
Elim(v..w,I);
```

and get a polynomial C which leads to the equation

$$C(p, q) := bp^2 - apq - bp - cq = 0. \tag{16}$$

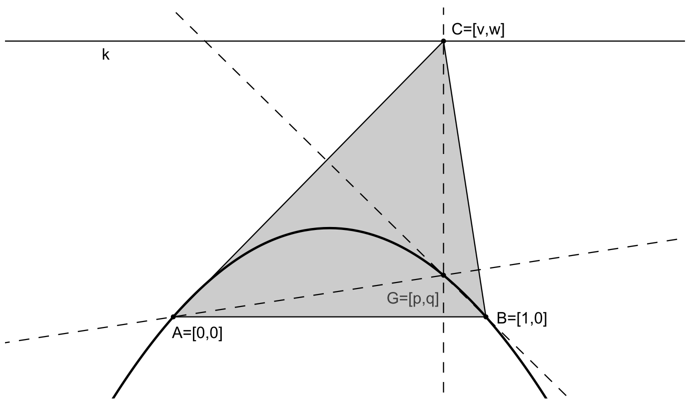

Figure 8: If C moves on $k \parallel AB$ then G lies on parabola

We can suppose that $(a, b) \neq (0, 0)$ since in this case the line k is not defined. Then (16) is the equation of a conic $C(p, q) = 0$.
The cases $k = h_{AB}$, $k = AC$, or $k = BC$ lead to singular conics which consist of two intersecting lines which are not depicted.

Considering regular conics we get *two* cases:

If $k \parallel AB$ the locus $C(p, q) = 0$ is a parabola with the vertex $[1/2, -b/(4c)]$ and a parameter $|c/(2b)|$, Fig. 8.

If $k \nparallel AB$ we obtain a hyperbola centered at $[-c/a, -b(a+2c)/a^2]$ with one asymptote perpendicular to AB and the second asymptote perpendicular to the line k through the intersection of the lines AB and k, Fig. 9.

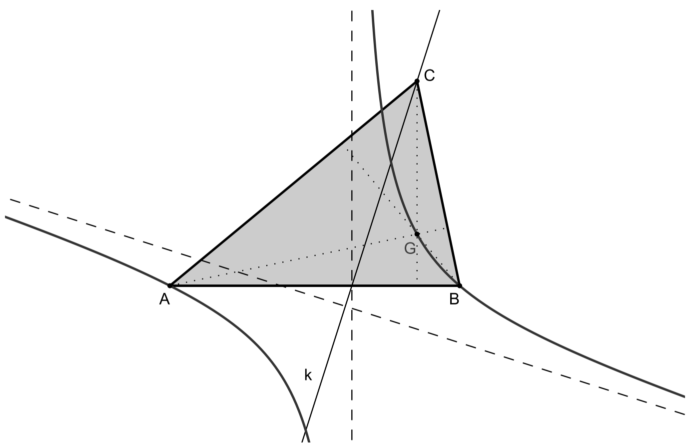

Figure 9: If C moves on $k \nparallel AB$ then G lies on hyperbola

In the given example we see that

- the use of DGS *does not suffice* to determine a curve exactly,

- the use of CAS was needed.

It would be helpful to find the locus classically.

The next example shows that the locus can be an algebraic curve of a higher degree.

Let ABC be a triangle with the given side AB and the vertex C on a circle k centered at B and radius $|AB|$. Find the locus of the orthocenter G of ABC if C moves on k.

First we construct the triangle ABC with the point C on the circle k in GeoGebra. Using a window Locus we construct the locus of the orthocenter G if C moves along k.
In the next step we derive the locus equation by CAS. We will use the same notation as in the previous case, Fig. 10. The situation is described by the following system of equations:

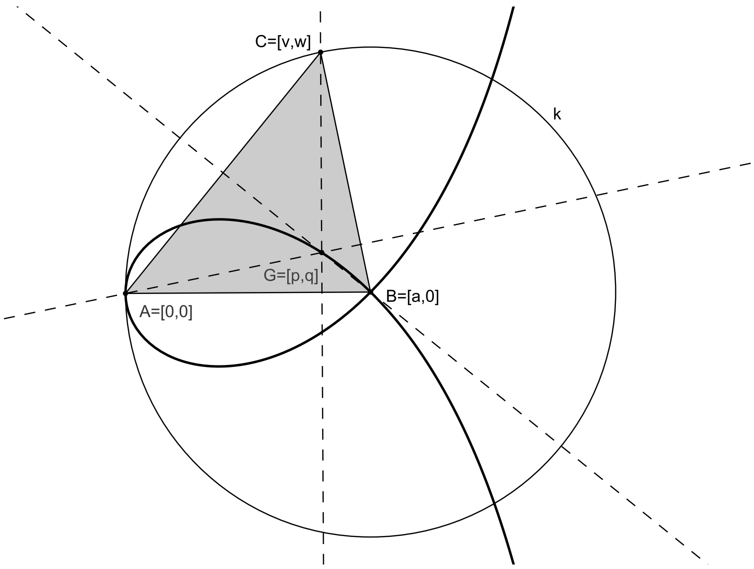

Figure 10: If C moves on k then G moves on a strophoid

$G \in h_{AB} \Rightarrow h_1 : p - v = 0,$

$G \in h_{BC} \Rightarrow h_2 : (v - a)p + wq = 0,$

$C \in k \Rightarrow h_3 : (v - a)^2 + w^2 - a^2 = 0.$

Elimination of v, w in the system $h_1 = 0$, $h_2 = 0$, $h_3 = 0$ gives in the program Epsilon

```
with(epsilon);
U:=[p-v,(v-a)*p+w*q,(v-a)^2+w^2-a^2]:
X:=[p,q,v,w]:
CharSet(U,X);
```

the equation

$$p^3 + pq^2 - 2p^2a - 2q^2a + pa^2 = 0 \tag{17}$$

which is the equation of a cubic curve called *strophoid* [14], Fig. 10.
The strophoid, or more exactly the right strophoid, has some interesting properties [4], [14], [5]. One of them is as follows:

Let k be a circle centered at S which is tangent to a given line AB at B. Let P be the intersection of the circle k and the line AS. If S moves along the perpendicular to AB at B then the locus of P is a

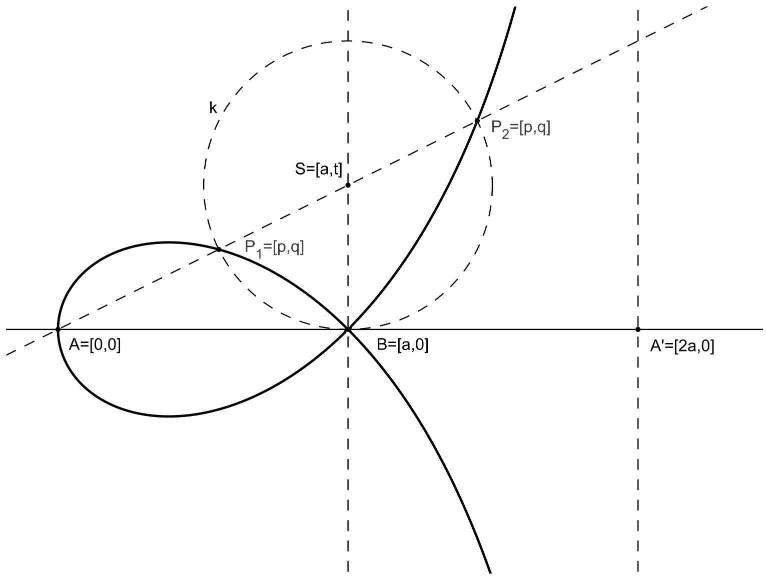

Figure 11: Definition of a strophoid

strophoid.

Let us derive the locus equation. We will use elimination of suitable variables in a coordinate system. Adopt a rectangular coordinate system so that $A = [0, 0]$, $B = [a, 0]$, $S = [a, t]$, $P = [p, q]$, Fig. 11. Algebraic description is as follows:

$P \in c \;\Rightarrow h_1 := (p - a)^2 + (q - t)^2 - t^2 = 0,$

P, S, A are collinear $\Rightarrow h_2 := \begin{vmatrix} p & q & 1 \\ a & t & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.$

Elimination of t in the system $h_1 = 0$, $h_2 = 0$ in Epsilon gives

```
with(epsilon);
U:=[(p-a)^2+(q-t)^2-t^2,p*t-q*a]:
X:=[p,q,a,t]:
CharSet(U,X);
```

the equation (17). We get the same locus as in the previous case.

Remark:

1. The property just proved is often used as the definition of a strophoid [4].

2. Observe that a strophoid is bounded from the right by the asymptote which is perpendicular to r and goes through the point $A' = [2a, 0]$, Fig. 11.

The fact that we obtained the same curve as the locus of *two* motions means that the strophoid has two following properties — it is the locus of the orthocenter of ABC when C moves on a given circle k_1, and it is also the locus of the intersection P of the line AS and the circle k_2. Let us prove classically that a strophoid, which is defined by intersections of a line with a circle, has the property shown in the example above:

For any point P of a strophoid which is given by the points A a B, the vertex C of a triangle ABC with the orthocenter P lies on the circle centered at B and radius $|AB|$, Fig. 12.

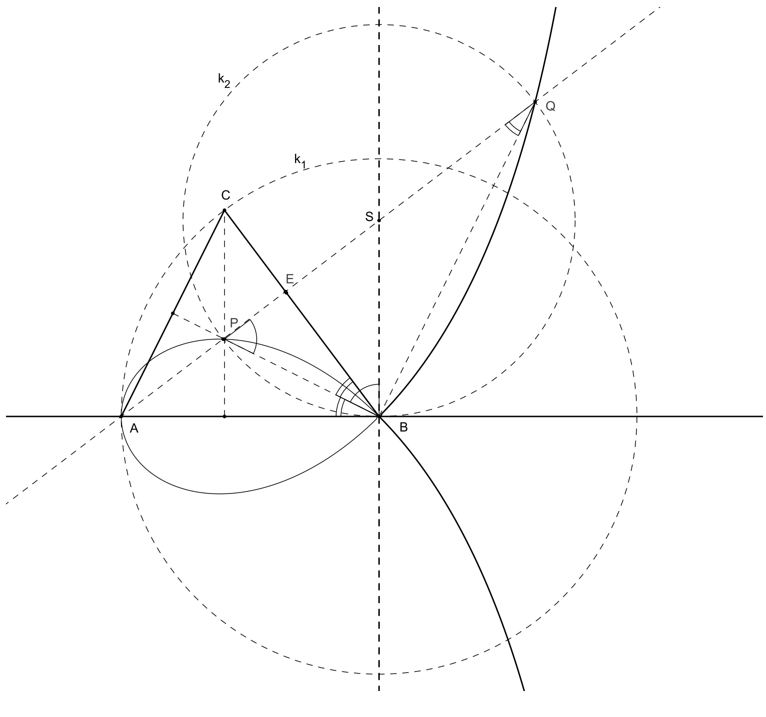

Figure 12: Classical proof

Let P be an arbitrary point of a strophoid which is the intersection of the line AS and the circle k_2. Construct the triangle ABC with the orthocenter P. We are to show that $C \in k_1$, Fig. 12.
By the theorem of Thales the triangle PBQ is right from which $|\angle BPQ| + |\angle PQB| = 90°$.
Right triangles PBE and PQB are similar which implies $|\angle PQB| = |\angle PBE|$.
As $|\angle ABS| = 90°$ and $|\angle BPQ| = |\angle PBS|$, then

$$|\angle ABP| = |\angle PBE|$$

and the triangle ABC is isosceles with $|AB| = |BC|$. Hence $C \in k_1$.

Now we will show another property of a strophoid. It is related to the well-known Steiner–Lehmus theorem [17]: *If a triangle has two internal angle bisectors of equal length, the triangle is isosceles.*

In [5] the modification of the Steiner–Lehmus theorem is studied. We will show:

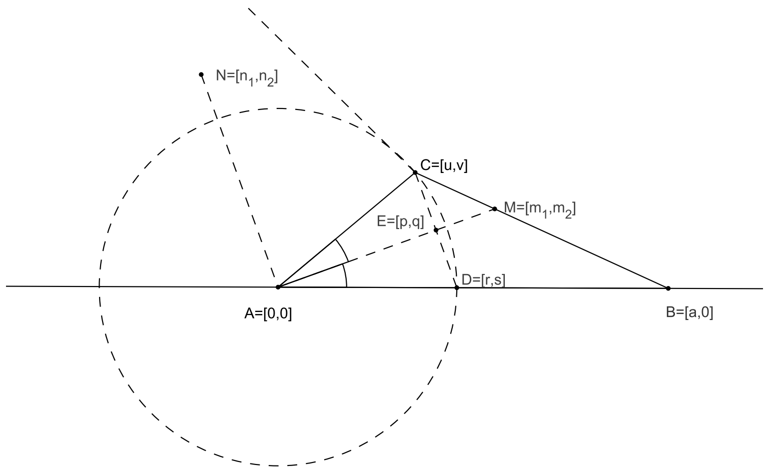

Figure 13: Internal and external angle bisectors at the vertex A are of equal length

Let ABC be a triangle with a fixed base AB. Then the locus of the vertex C such that internal and external angle bisectors at the vertex A are of equal length is a strophoid given by the vertices A and B.

The locus can be find in the following way. Choose a system of coodinates so that $A = [0,0]$, $B = [a,0]$, $C = [u,v]$, $M = [m_1,m_2]$, $N = [n_1,n_2]$, $D = [r,0]$, $E = [p,q]$, Fig. 13. Then:

$|AC| = |AD| \Rightarrow h_1 := u^2 + v^2 - r^2 = 0,$

E is the center of $AD \Rightarrow h_2 := u + r - 2p,\ h_3 := v - 2q = 0,$

A, E, M are collinear $\Rightarrow h_4 := m_1q - m_2p = 0,$

B, M, C are collinear $\Rightarrow h_5 := um_2 + va - m_2a - m_1v = 0,$

$AN \perp AM \Rightarrow h_6 := m_1n_1 + m_2n_2 = 0,$

B, N, C are collinear $\Rightarrow h_7 := un_2 + va - n_2a - n_1v = 0,$

$|AM| = |AN| \Rightarrow h_8 := m_1^2 + m_2^2 - n_1^2 - n_2^2 = 0.$

Elimination of dependent variables $r, p, q, m_1, m_2, n_1, n_2$ in the ideal $I = (h_1, h_2, \ldots, h_8)$ in CoCoA

```
Use R::=Q[u,v,a,r,p,q,m[1..2],n[1..2]];
I:=Ideal(u^2+v^2-r^2,u+r-2p,v-2q,m[1]q-m[2]p,um[2]+va-m[2]a-m[1]v,
m[1]n[1]+m[2]n[2],un[2]+va-n[2]a-n[1]v,m[1]^2+m[2]^2-n[1]^2-n[2]^2);
Elim(r..n[2],I);
```

gives

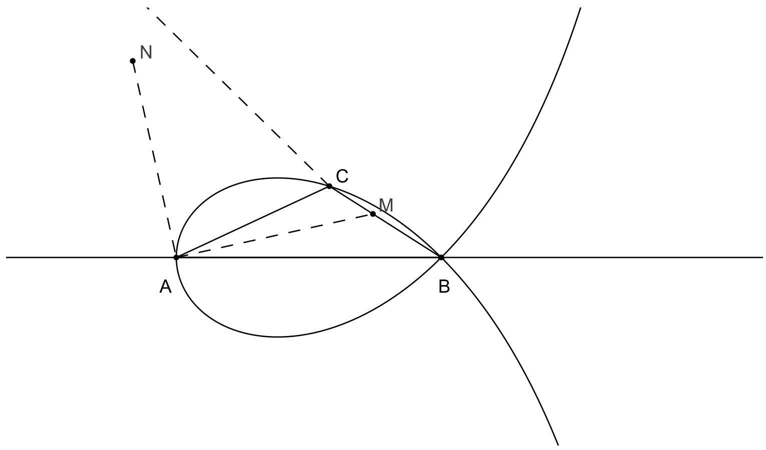

Figure 14: If $|AM| = |AM|$ then the locus of C is a strophoid

$$C(u,v) := (u^2+v^2)(u-2a)+ua^2 = 0$$

which is the equation of a strophoid, Fig. 14.

Now we have to show that for *any* point C of the strophoid the bisectors at the vertex A are of equal length. To do this we express the normal form NF of the polynomial h_8 with respect to the ideal $J = (h_1, h_2, \dots, h_7, av-1, C)$, where we excluded the cases $a = 0$, i.e. $A = B$ and $v = 0$, i.e A, B, C are collinear. Entering in CoCoA

```
Use R::=Q[u,v,a,r,p,q,m[1..2],n[1..2],t,s];
J:=Ideal(u^2+v^2-r^2,u+r-2p,v-2q,m[1]q
-m[2]p,um[2]+va-m[2]a-m[1]v,m[1]n[1]+m[2]n[2],un[2]+va-n[2]a-n[1]v,
(u^2 + v^2)(u - 2a) + ua^2,av-1);
NF(m[1]^2+m[2]^2-n[1]^2-n[2]^2,J);
```

we get the result $NF = 0$. This means that the polynomial h_8 belongs in the ideal J and the statement is proved.

Let us show classically that a strophoid, which is defined above by intersections of a line with a circle, has the following property:

For any point C of a strophoid which is given by the points A a B, internal and external angle bisectors at the vertex A of a triangle ABC are of equal length, Fig. 15.

From $\triangle ABM$ the equality

$$\omega + \beta = \alpha$$

follows. Similarly from $\triangle ABX$ we get

$$2(\omega+\beta) = 90^\circ.$$

Then $\alpha = 45^\circ$ and $\triangle AMN$ is isoceles.

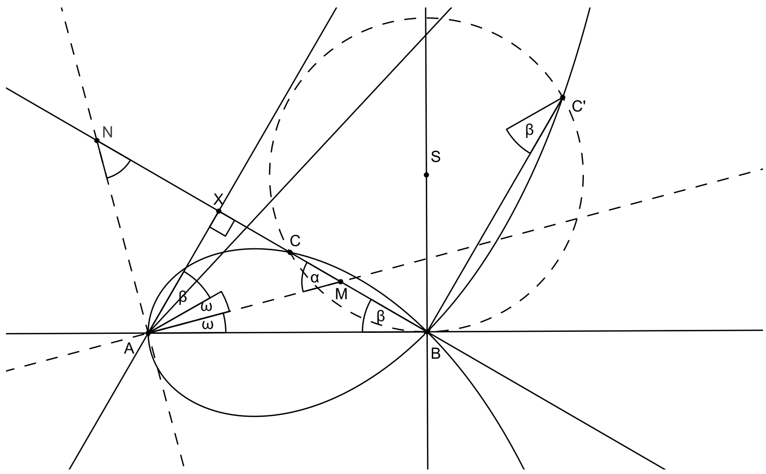

Figure 15: Internal and external angle bisectors at A are of equal length—classical proof

3.2 Loci in space

Next we will show an example on searching locus equation in space. This example is based of the well-known Simson–Wallace theorem which reads: *Let ABC be a triangle and P a point of the circumcircle of ABC. Then the feet of perpendiculars K, L, M from P onto the sides of ABC lie on a straight line.*

See [2] for details.

Now the question arises. What is the analogy of this theorem in space? Instead of a triangle we can take a tetrahedron $ABCD$. And what is the analogy of a circumcircle of a triangle in the case of a tetrahedron? Is it a sphere? Most students said "yes." The answer is given in the solution of the following problem [8], [11]:

Let K, L, M, N be the feet of perpendiculars dropped from a point P onto the faces BCD, ACD, ABD, ABC of a tetrahedron $ABCD$. What is the locus of P such that K, L, M, N are complanar?

Choose a rectangular system of coordinates so that $A = [0, 0, 0]$, $B = [1, 0, 0]$, $C = [b, c, 0]$, $D = [d, e, f]$, $K = [k_1, k_2, k_3]$, $L = [l_1, l_2, l_3]$, $M = [m_1, m_2, m_3]$, $N = [n_1, n_2, n_3]$, $P = [p, q, r]$, Fig. 16.

The following relations describe the points K, L, M, N:

$PK \perp BCD \Rightarrow$
$h_1 := (b-1)(p-k_1) + c(q-k_2) = 0,\ h_2 := (d-1)(p-k_1) + e(q-k_2) + f(r-k_3) = 0,$

$K \in BCD \Rightarrow h_3 := -cf - ek_3 + fk_2 + ck_3 + cfk_1 + bek_3 - cdk_3 - bfk_2 = 0,$

$PL \perp ACD \Rightarrow$
$h_4 := b(p-l_1) + c(q-l_2) = 0,\ h_5 := d(p-l_1) + e(q-l_2) + f(r-l_3) = 0,$

$L \in ACD \Rightarrow h_6 := cfl_1 + bel_3 - cdl_3 - bfl_2 = 0,$

$PM \perp ABD \Rightarrow$

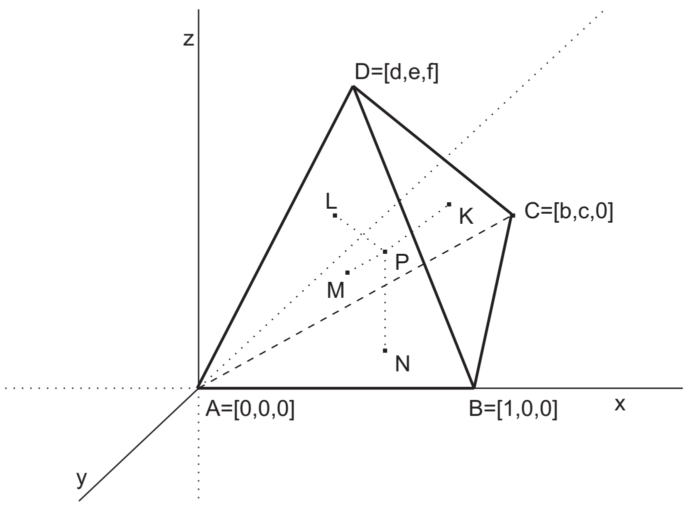

Figure 16: Generalization of the Simson–Wallace theorem on a tetrahedron

$h_7 := p - m_1 = 0,\ h_8 := d(p - m_1) + e(q - m_2) + f(r - m_3) = 0,$

$M \in ABD \Rightarrow h_9 := em_3 - fm_2 = 0,$

$PN \perp ABC\ \Rightarrow h_{10} := p - n_1 = 0,\ h_{11} := b(p - n_1) + c(q - n_2) = 0.$

Conclusion is of the form:

$$K, L, M, N \text{ are complanar} \quad \Rightarrow \quad h_{12} := \begin{vmatrix} k_1 & k_2 & k_3 & 1 \\ l_1 & l_2 & l_3 & 1 \\ m_1 & m_2 & m_3 & 1 \\ n_1 & n_2 & 0 & 1 \end{vmatrix} = 0.$$

Elimination of variables $k_1, k_2, k_3, l_1, l_2, l_3, m_1, m_2, m_3, n_1, n_2$ in the system of polynomials $h_1, h_2, \ldots, h_{12}$ in the program Epsilon

```
with(epsilon);
U:=[(b-1)*(p-k[1])+c*(q-k[2]),(d-1)*(p-k[1])+e*(q-k[2])+f*(r-k[3]),-c*f-e*k
+f*k[2]+c*k[3]+c*f*k[1]+b*e*k[3]-c*d*k[3]-b*f*k[2],b*(p-l[1])+c*(q-l[2]),
d*(p-l[1])+e*(q-l[2])+f*(r-l[3]),c*f*l[1]+b*e*l[3]-c*d*l[3]-b*f*l[2],p-m[1]
d*(p-m[1])+e*(q-m[2])+f*(r-m[3]),e*m[3]-f*m[2],p-n[1],b*(p-n[1])+c*(q-n[2])
k[1]*l[2]*m[3]-k[1]*m[2]*l[3]+k[1]*n[2]*l[3]-k[1]*n[2]*m[3]-l[1]*k[2]*m[3]+
l[1]*m[2]*k[3]-l[1]*n[2]*k[3]+l[1]*n[2]*m[3]+m[1]*k[2]*l[3]-m[1]*l[2]*k[3]+
m[1]*n[2]*k[3]-m[1]*n[2]*l[3]-n[1]*k[2]*l[3]+n[1]*k[2]*m[3]-n[1]*l[2]*m[3]
+n[1]*l[2]*k[3]-n[1]*m[2]*k[3]+n[1]*m[2]*l[3]]:
X:=[b,c,d,e,f,p,q,r,k[1],k[2],k[3],l[1],l[2],l[3],m[1],m[2],m[3],n[1],n[2]]
CharSet(U,X);
```

gives the equation[4]

[4] In CoCoA which is based on Gröbner bases computation we need to use successive elimination to obtain the same result.

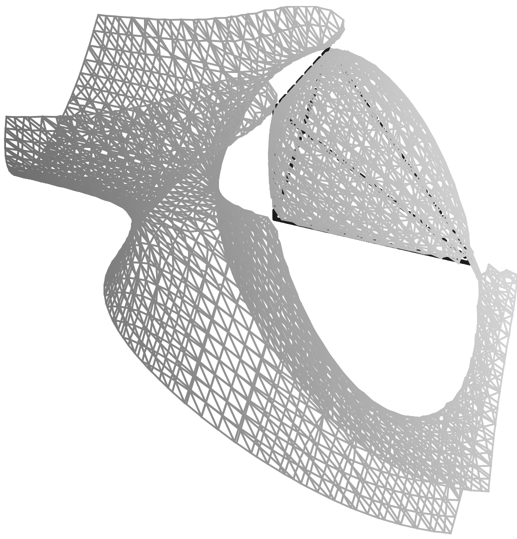

Figure 17: Cubic surface $p^2q + pq^2 + p^2r + q^2r + pr^2 + qr^2 - pq - pr - qr = 0$

$C(p, q, r) := c^2f^2p^2q + cf(f^2 + e^2 - ce)p^2r - cf^2(2b - 1)pq^2 - cf^2(2d - 1)pr^2 + 2cef(b - d)pqr + bf^2(b-1)q^3 + f(cf^2 - b^2e + be - cd + cd^2)q^2r + f^2(b^2 - 2ce - b + c^2)qr^2 + f(cd^2 - cd - eb^2 - ec^2 + e^2c + be)r^3 - c^2f^2pq + cf(ce - e^2 - f^2)pr + bcf^2q^2 + f(2bcde - 2bce - c^2d^2 - b^2e^2 + be^2 + c^2d - b^2f^2 - c^2f^2 + bf^2)qr + (b^2ef^2 - bef^2 + c^2d^2e - c^2de + c^2ef^2 + bce^2 + cde^2 + b^2e^3 - 2bcde^2 - be^3 + cdf^2)r^2 = 0.$

It is the equation of a cubic surface $C(p, q, r) = 0$ which is known as a Cayley surface [3].
We see that the locus is not a sphere as it could seem from the Simson–Wallace theorem in a plane.

For $b = 0$, $c = 1$, $d = 0$, $e = 0$ and $f = 1$ we get a cubic surface

$$p^2q + pq^2 + p^2r + q^2r + pr^2 + qr^2 - pq - pr - qr = 0 \tag{18}$$

which is depicted in Fig. 17.
The cubic surface (18) has four singular points at vertices $A = [0, 0, 0]$, $B = [1, 0, 0]$, $C = [0, 1, 0]$, $D = [0, 0, 1]$. It contains all six edges of a tetrahedron $ABCD$.

Conclusion

Derivation of new statements brings new quality into mathematics and also in mathematics education. Whereas in the past we explored at schools loci such as lines, segments, circles, conics, spheres and quadrics, nowadays the situation is changing. Due to computers and appropriate software we can investigate more complicated loci both in a plane and in space. By dynamic geometry systems we are able to demonstrate them whereas by computer algebra systems we do exact mathematical proofs.
This makes us possible to meet less known algebraic curves of higher order and study them and even discover new properties. Similar approach can be applied by investigation of loci in space as we could see in the last example.
Technique mentioned above should not exclude the use of classical methods if it is possible.

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